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UNIFORM EMBEDDINGS OF HYPERBOLIC GROUPS IN HILBERT SPACES

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ABSTRACT

We construct uniform embeddings of the Cayley graphs of hyperbolic groups and cyclic extensions of torsion-free small cancellation groups in Hilbert spaces.

Introduction

In [Bo] J. Bourgain has shown that in superreflexive Banach spaces there is no bi-Lipschitz embedding of a regular tree. In this paper we discuss a weaker notion of embedding, a uniform one.

Let A and B be metric spaces and let $e : A \to B$ be an embedding. e is called uniform if it is Lipschitz and there exists a function φ so that $\lim_{t\to\infty} \varphi(t) = \infty$ and for all $x, y \in A$:

$$d_B(e(x), e(y)) \geq \varphi(d_A(x, y)).$$

A uniform embedding of the Cayley graph of a f.p. group into a Hilbert space plays an essential role in the work of A. Connes, M. Gromov and H. Moscovici [Co-Gr-Mo] around the Novikov conjecture.

By modifying the construction of canonical representatives which was introduced in [Ri-Se] we construct uniform embeddings for the Cayley graphs of hyperbolic groups and of cyclic extensions of small cancellation groups satisfying condition $C'\left(\frac{1}{8}\right)$.

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In section 1 we bring some preliminaries, and modify the construction of canonical representatives introduced in [Ri-Se] for the present purposes, in section 2 we prove certain stability properties of the construction and in section 3 we use this stability to get uniform embeddings of hyperbolic groups in Hilbert spaces. In section 4 we apply results already achieved in [Ri-Se] for torsion-free small cancellation groups, to get uniform embeddings of cyclic extensions of such groups in Hilbert spaces.

The whole problem of uniform embeddings of groups in Hilbert spaces was introduced to me by Prof. M. Gromov. I am greatly indebted to him for some valuable discussions around these questions. On the concept of canonical representatives I've learned from my advisor Prof. E. Rips. Although we do not use the proposed terminology, we believe they should be called Rips' canonical representatives.

1. Preliminaries

To construct our embeddings for hyperbolic groups we need to modify some of the definitions and constructions introduced for solving equations in torsion-free hyperbolic groups [Ri-Se].

Let $\Gamma =< G|R >$ be a δ-hyperbolic group with a Cayley graph X. A μ-local geodesic in X is a path $f:[a,b] \to X$ satisfying :

$$\text{length}(f([a',b'])) \le \mu \Rightarrow \text{length}(f([a',b'])) = |f(a') - f(b')|.$$

A λ-local quasigeodesic in X is a path $f:[a,b] \to X$ satisfying

$$\text{length}(f([a',b'])) \le 1000\delta\lambda \Rightarrow \text{length}(f([a',b'])) \le \lambda|f(a') - f(b')|.$$

Definition 1.1: Let BL_r be the ball of radius r in the Cayley graph X. Let $\mu_0 = 20000\delta^2(1+\log(10\delta))$. A vertex $v \in X$ belongs to Zone k of X, zone $(v) = k$, if $v \in BL_{k\mu_0} \setminus BL_{(k-1)\mu_0}$. ∎

Definition 1.2: Let $B_0 = 0$, $B_m = (5\cdot 2^m)^{5\cdot 2^{m-1}}$. A sequence $\underline{b} = \{b_j\}_{j=1}^{\infty}$, $1 \le b_j \le 10$ is called prefix-avoiding if for every $1 \le m < \infty$ and every n;

$$\frac{B_{m-1}}{5\cdot 2^m} \le n \le \frac{B_m}{5\cdot 2^m} - 1$$

the sequence $b_{5\cdot 2^m\cdot n+1}, \dots, b_{5\cdot 2^m(n+1)}$ does not appear as a consecutive subsequence of the prefix $b_1, \dots, b_{5\cdot 2^m\cdot n}$. ∎

The following definition is similar to definition 1.1 of [Ri-Se], modified for our purposes:

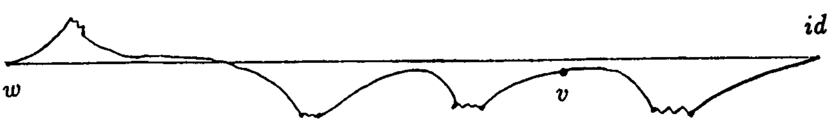
Definition 1.3: Let $\underline{b}^T = \{b_j^T\}_{j=1}^{\infty}$ be a prefix-avoiding sequence. A T prefix-avoiding coarse piecewise geodesic $f : [a,b] \to X$, $a = c_1 \leq d_1 \leq c_2 \cdots \leq d_q = b$, is a 10δ-local quasigeodesic so that $f([c_i, d_i])$ is a μ_0-local geodesic and :

$$|zone(f(d_1)) - zone(f(a))| \geq 10.$$
$$|zone(f(d_i)) - zone(f(d_{i-1}))| \geq 3b_i^T \qquad 2 \leq i \leq q-1.$$
$$\text{length}(f([d_i, c_{i+1}])) \leq 2\delta \qquad 1 \leq i \leq q-1.$$

A restriction $f|_{[c_i,d_i]}$ is called sub-local geodesic and a restriction $f|_{[d_i,c_{i+1}]}$ is called bridge.

Definition 1.4: ([Ri-Se], 2.1). Let $w \in \Gamma$ be given. A vertex $v \in X$ is called an elector of w with respect to a criterion T if there exists a map $f : [a,b] \to X$ through v so that:

(i) $f(a) = id; f(b) = w$.
(ii) Let $h : [a, f^{-1}(v)] \to X$ be given by $h(a+t) = f(f^{-1}(v) - t)$. Then h and $f|_{[f^{-1}(v),b]}$ are T prefix-avoiding coarse piecewise geodesics.
(iii) v lies on a μ_0-local geodesic e, where e is the union of the first μ_0-sub local geodesics of the prefix-avoiding coarse piecewise geodesics h and $f|_{[f^{-1}(v),b]}$



The set of all electors with respect to a criterion T is called T-cylinder of $w, C_T(w)$.

LEMMA 1.5: *([Ri-Se], 1.2). Let $\gamma = [id, w]$ be a geodesic segment, let $v \in C_T(w)$ and let $f : [a,b] \to X$; $f(a) = id$; $f(b) = w$ be a map through v satisfying the conditions of the above definition. Let $g : [c,d] \to X$ be a sub-local geodesic of f and let $z \in g$ satisfy:*

$$\min(|z - f(c)|, |z - f(d)|) \geq 1100\delta^2(1 + \log(10\delta))$$

then z is 2δ-close to γ.

LEMMA 1.6: *With the above notations let $g_i : [c_i, d_i] \to X$ be the i-th sub local geodesic of $f|_{[f^{-1}(v),b]}$ and suppose $|zone(f(d_i)) - zone(f(d_{i-1}))| > 3b_i^T$. Let $d_{i-1} < t < d_i$ be the first value for which $|zone(f(t)) - zone(f(d_{i-1}))| = 3b_i^T$, and let z_0 be one of the closest points to $f(t)$ on γ. Then if we modify f by setting $d_i = t$; $f(c_{i+1}) = z_0$; $f(d_{i+1}) = w$ we have a new map through w satisfying the conditions of Definition 1.4.*

The lemma is a modified version of lemma 1.3 of [Ri-Se], the proofs are identical and therefore we prefer to skip it.

Definition 1.7: ([Ri-Se], 3.1). Let $\gamma = [id, v]$; $v \in X$ be a geodesic segment in the Cayley graph X. A geodesic not shorter than γ in a 2δ-neighborhood of γ is called a channel of γ. The μ_0-capacity of $\Gamma, Ca(\mu_0)$, is the maximal number of different channels of a geodesic with length μ_0. A loose bound on $Ca(\mu_0)$ is $2^{v_{2\delta}\mu_0}$, where $v_{2\delta}$ is the volume of a ball with radius 2δ in X. ∎

2. Stability Properties of Cylinders

To get uniform embeddings of the Cayley graph X in a Hilbert space, we need the T-cylinders to have certain global stability properties, i.e., for any two close words $w_1, w_2 \in \Gamma$ we want the symmetric difference between their cylinders $C_T(w_1)\Delta C_T(w_2)$ to be controlled. Unfortunately for hyperbolic groups we are not able to get cylinders with the quality we got for small cancellation groups in [Ri-Se] (see section 4 below), i.e., a global bound for the symmetric difference in terms of the distance $|w_1 - w_2|$. However, the following theorem turns to be sufficient for uniform embeddings:

THEOREM 2.1: *Let $w_1, w_2 \in \Gamma$; $|w_1 - w_2| = 1$. For every $1 \leq m < \infty$ let*

$$p(m) = \left\lfloor log_2 \frac{B_m}{200} \right\rfloor.$$

Let $\ell_{B_m}^T = 3 \cdot \sum_{j=1}^{B_m} b_j^T$, and D_m be the set:

$$D_m = \{C_T(w_1) \backslash C_T(w_2)\} \cap \left\{ v \in \Gamma | \ell_{B_m}^T \leq zone(w_1) - zone(v) \leq \ell_{Bp(m)}^T \right\},$$

then $|D_m| \leq 2 \cdot Ca(\mu_0) \cdot v_{2\delta} \cdot \mu_0$

Proof: Let $\gamma = [id, w_1]$ be a geodesic segment and let $\tau_1 = [v_1, v_2]$; $\tau_2 = [v_3, v_4]$; $\tau_3 = [v_5, v_6]$ be subsegments of γ so that:

$[v_5, v_6] = \gamma \cap \{v \in X | zone(v) = zone(w_1) - \ell^T_{B_m} + 100\}$.
$[v_3, v_4] = \gamma \cap \{v \in X | zone(v) = zone(w_1) - \ell^T_{B_m} + 102\}$.
$[v_1, v_2] = \gamma \cap \{v \in X | zone(w_1) - \ell^T_{B_m} + 103 \leq zone(v) \leq zone(w_1) - \frac{1}{10}\ell^T_{B_m}\}$.

Recall, the electors are picked according to the existence of a pair of prefix-avoiding coarse piecewise geodesics through them. Let $u_1, u_2 \in C_T(w_1) \backslash C_T(w_2)$;

$$\ell^T_{B_m} \leq zone(w_1) - zone(u_i) \leq \ell^T_{B_{p(m)}} \quad (i = 1, 2).$$

We have $\beta_i : [a, b] \to X$, T prefix-avoiding coarse piecewise geodesics so that $\beta_i(a) = u_i$; $\beta_i(b) = w_1$.

Claim 2.2: Suppose β_1, β_2 occupy the same channel W of either τ_2 or τ_3, then $zone(u_1) = zone(u_2)$.

Proof: Let ν_1, ν_2 be the μ_0-sub local geodesics of β_1, β_2 which pass through W. Clearly W can not be τ_2 or τ_3 themselves, otherwise we can modify β_i by continuing through γ and after crossing 20 zones make a bridge to a geodesic between the identity and w_2, so we have $u_i \in C_T(w_2)$ (see Lemma 1.6), a contradiction. ∎

Let ex_i be the number of zones ν_i has to get through after passing through W and let $t^i_1, t^i_2, \ldots$ be the lengths in zones of each of the μ_0-sub local geodesics of β_i afterwards.

LEMMA 2.3: *$ex_1 = ex_2$ and $t^1_s = t^2_s$ for all μ_0-sub local geodesics over τ_1 (i.e. in a $1100\delta^2(1 + log(10\delta))$ neighbourhood of τ_1). Moreover the t^i_s are identical with the corresponding $b^T_{j(i,s)}$; $j(i, s) = s + s^i_0$ from the T prefix-avoiding sequence.*

Proof: Suppose $ex_1 = ex_2$ and let $\tilde{s}$ be the first index for which $t^1_{\tilde{s}} \neq t^2_{\tilde{s}}$ (say $t^1_{\tilde{s}} < t^2_{\tilde{s}}$). Then we can modify β_1 by continuing through β_2 after passing through W and in the $\tilde{s}$ μ_0-sub local geodesic of β_2 make a bridge to γ (according to lemma 1.6) and then a bridge to a geodesic from the identity to w_2 and we have $u_1 \in C_T(w_2)$. Clearly if $t^1_s > b^T_{j(1,s)}$ we can modify β_1 by making a bridge after crossing $b^T_{j(1,s)}$ zones with the s μ_0-sub local geodesic of β_1 and have $u_1 \in C_T(w_2)$. If $ex_1 > ex_2$ we can modify β_2 by continuing through β_1 after passing through W and then make a bridge to γ and have $u_2 \in C_T(w_2)$. ∎

Now, τ_1 crosses more than $\frac{4}{5}\ell^T_{B_m}$ zones, and so at least $\frac{4\ell^T_{B_m}}{150}\mu_0$-sub local geodesics. By our assumptions $zone(w_1) - zone(u_i) < \ell^T_{B_{p(m)}}$, for which we have the following simple fact:

LEMMA 2.4: *In a prefix-avoiding sequence, there are no identical subsequences of consecutive elements of length $3 \cdot 5 \cdot 2^m$ in a prefix of length B_m.*

Proof: An easy exercise which we leave for the reader. ∎

We are interested in the sequences for the u_i $(i = 1, 2)$ which are included in a prefix of length $B_{p(m)}$. So by the lemma every subsequence of consecutive elements of length

$$3 \cdot 5 \cdot 2^{p(m)} \leq \frac{3 \cdot B_m}{40} \leq \frac{\ell^T_{B_m}}{40} < \frac{\ell^T_{B_m} \cdot 4}{150}$$

is disjoint and so by Lemma 2.3 $s^1_0 = s^2_0$, i.e., ν_1 and ν_2 the sub-local geodesics of β_1 and β_2 passing through W, have the same index in the prefix-avoiding sub local geodesics β_1 and β_2.

Suppose $zone(u_2) < zone(u_1)$. The number of μ_0-sub local geodesics in β_1 and β_2 before passing through W is identical, and so there must be a sub local geodesic g_{j_0} of β_2 which crosses more zones than the sub local geodesic with the same index in β_1, and therefore more than $3b^T_{j_0}$, the minimum required for a prefix-avoiding coarse piecewise geodesic. Clearly we can modify β_2 by making a bridge to γ (Lemma 1.6), after passing $3b^T_{j_0}$ zones in g_{j_0} and get $u_2 \in C_T(w_2)$, a contradiction. ∎

Having the claim, the theorem follows easily since the number of electors of $C_T(w_1)$ in a zone is bounded by $\mu_0 \cdot v_{2\delta}$ (every elector is 2δ-close to γ), and the number of channels over τ_2 and τ_3 is bounded by $Ca(\mu_0)$ for each. Note that by the conditions on prefix-avoiding coarse piecewise geodesics, it must occupy some channel over either τ_2 or τ_3 if its starting point is according to the conditions of the theorem. ∎

3. Uniform Embeddings of Hyperbolic Groups

Let H be a Hilbert space and let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of H. At each vertex v of the Cayley graph X we place a distinct basis element denoted by e_v.

Definition 3.1: Let Γ be a δ-hyperbolic group and let $\underline{b}^T = \{b^T_j\}$ be a prefix-avoiding subsequence. A T embedding $U_T : \Gamma \to H$ is defined by:

$$U_T(w) = \sum_{v \in C_T(w)} \frac{1}{|v-w|^{\alpha}} e_v, \qquad 0 < \alpha \leq \frac{1}{2}.$$

∎

LEMMA 3.2: *U_T is Lipschitz.*

Proof: Let $\{k_s\}_{s=0}^{\infty}$ be the sequence given by $k_0 = 1$; $k_s = p(k_{s-1})$. From the definition of the function p it is not hard to see that $k_s \geq e^s$. (In fact it grows much faster, but for us it is enough.)

Let $w_1, w_2 \in \Gamma$; $|w_1 - w_2| = 1$. We have by Theorem 2.1:

$$\begin{aligned}
|U_T(w_1) - U_T(w_2)|^2 \leq & \left| \sum_{v \in C_T(w_1) \setminus C_T(w_2)} \frac{1}{|v - w_1|^{\alpha}} e_v \right|^2 \\
& + \left| \sum_{v \in C_T(w_2) \setminus C_T(w_1)} \frac{1}{|v - w_2|^{\alpha}} e_v \right|^2 \\
& + \left| \sum_{v \in C_T(w_1) \cap C_T(w_2)} \left(\frac{1}{|v - w_1|^{\alpha}} - \frac{1}{|v - w_2|^{\alpha}} \right) e_v \right|^2 \\
\leq & \; 2\ell_{B_1}^T v_{2\delta} + 2 \sum_{s=0}^{\infty} \frac{2Ca(\mu_0) \cdot \mu_0 \cdot v_{2\delta}}{(\ell_{k_s}^T)^{2\alpha}} \\
& + v_{2\delta} \sum_{d=1}^{\infty} \left[\frac{1}{(d+1)^{\alpha}} - \frac{1}{d^{\alpha}} \right]^2 \\
\leq & \; f(\delta, v_{2\delta}, \alpha).
\end{aligned}$$

∎

THEOREM 3.3: *U_T is uniform.*

Proof: Let $w_1, w_2 \in \Gamma$. Since electors are 2δ-close to the corresponding geodesics, we have for at least one of the w_i, say w_1:

$$C_T(w_2) \cap \left\{ v \in X \;\middle|\; |w_1 - v| < \frac{|w_1 - w_2|}{2} - 20\delta \right\} = \phi.$$

But all the vertices which lie on a geodesic from the identity to w_1 and located at distance bigger than $10\mu_0$ from both w_1 and the identity are necessarily electors. Therefore:

$$|U_T(w_2) - U_T(w_1)| \geq \left[\sum_{d=10\mu_0}^{\frac{|w_1 - w_2|}{2} - 20\delta} \frac{1}{d^{2\alpha}} \right]^{1/2} = \varphi(|w_1 - w_2|, \delta, \alpha)$$

where for fixed δ and α; $0 < \alpha \leq \frac{1}{2}$ we have

$$\lim_{|w_1 - w_2| \to \infty} \varphi(|w_1 - w_2|, \delta, \alpha) = \infty.$$

∎

4. Cyclic Extensions of Small Cancellation Groups

For small cancellation groups satisfying condition $C'(\frac{1}{8})$ our cylinders in [Ri-Se] are globally stable. In particular we got the following theorem for torsion-free ones:

THEOREM 4.1 (([Ri-Se], 4.3)): *Let $\Gamma = < G|R >$ be a torsion-free group, satisfying $C'(\frac{1}{8})$. There exist canonical representatives $\theta_T : \Gamma \to F(G)$ so that if $w_1, w_2, w_3 \in \Gamma$; $w_1 w_2 w_3 = 1$ then there exist c_i, $f^{ij}, y_0^i, y_1^i \in F(G)$ so that :*

(i)
$$\begin{aligned}
\theta_T(w_1) &= y_0^1\ f^{11}\ y_1^1\ c_1\ \left(y_1^2\right)^{-1} f^{21}\ \left(y_0^2\right)^{-1},\\
\theta_T(w_2) &= y_0^2\ f^{22}\ y_1^2\ c_2\ \left(y_1^3\right)^{-1} f^{32}\ \left(y_0^3\right)^{-1},\\
\theta_T(w_3) &= y_0^3\ f^{33}\ y_1^3\ c_3\ \left(y_1^1\right)^{-1} f^{13}\ \left(y_0^1\right)^{-1}.
\end{aligned}$$

(ii) $\quad c_1c_2c_3;\ f^{11}f^{13};\ f^{21}f^{22};\ f^{32}f^{33}$ *are elements in* $< R >^{F(G)}$.

(iii)
$$\begin{aligned}
length(c_i) &\leq 80(\mu_0 + \delta)v_{2\delta}(20\delta + 1),\\
length(f^{ij}) &\leq 40(\mu_0 + \delta)v_{2\delta}(20\delta + 1),
\end{aligned}$$

where

$$\mu_0 = 20000r^2(1 + \log(10r)) \quad \textit{and} \quad r = \max(\delta, \max_{r_i \in R} \mathrm{length}(r_i)).$$

Remark: With minor modifications the theorem remains valid for $C'(\frac{1}{7})$ groups with no 2-torsion. ∎

Let $\Gamma = < G|R >$ be as above and let M be a cyclic extension of Γ, so that we have:

$$1 \to Z \to M \to \Gamma \to 1.$$

Let z be a generator of the normal cyclic subgroup Z and let $x_1, \ldots, x_k \in M$ so that x_i is mapped to $g_i \in G$, the generating set for the small cancellation group Γ under the homomorphism $\Psi : M \to \Gamma$. Let X_M be the Cayley graph of M with the generators $\{z, x_1, \ldots, x_k\}$. Let $M_0 \triangleleft M$ be the centralizer of Z in M and $\Gamma_0 \triangleleft \Gamma$ its image in Γ. W.l.o.g. $x_1 \notin M_0$ if the extension is not central.

Definition 4.2: For each $w \in \Gamma$ let $\lambda_T(w) \in M$ be the element obtained from $\theta_T(w) \in F(G)$ by substituting each g_i with x_i. Clearly, each element $m \in M$ can be represented uniquely as:

$$m = z^{\nu(m)} \lambda_T(\Psi(m x_1^{-\epsilon(m)})) x_1^{\epsilon(m)}$$

where $\epsilon(m)$ is 0 if $m \in M_0$ and 1 otherwise. ∎

Let H be a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$. To each vertex of the Cayley graph $v \in X_\Gamma$ we adjoin a distinct element from the set $\{e_i\}_{i=2}^{\infty}$ denoted by e_v. Let $w \in \Gamma$ be given. We denote by $V(w)$ the set of all vertices $v \in X_\Gamma$ which correspond to prefixes of the canonical representative $\theta_T(w)$ (i.e. all the vertices on the canonical path from the identity to w).

Our embedding $U_T : M \to H$ will send an element represented by the form described in Definition 4.2 to the vector:

$$U_T(m) = \nu(m) e_1 + \sum_{v \in V(\Psi(m x_1^{-\epsilon(m)}))} e_v.$$

LEMMA 4.3: *There exists a constant q (depending on M and the choice of $x_1, \ldots, x_k$), so that for all triples $w_1, w_2, w_3 \in \Gamma$; $w_1 w_2 w_3 = 1$,*

$$\lambda_T(w_1)\lambda_T(w_2)\lambda_T(w_3) = z^t \quad \textit{where} \quad |t| < q.$$

Proof: By Theorem 4.1 we have:

$$\begin{aligned} \theta_T(w_1) &= y_0^1 \, f^{11} \, y_1^1 \, c_1 \left(y_1^2\right)^{-1} f^{21} \left(y_0^2\right)^{-1}, \\ \theta_T(w_2) &= y_0^2 \, f^{22} \, y_1^2 \, c_2 \left(y_1^3\right)^{-1} f^{32} \left(y_0^3\right)^{-1}, \\ \theta_T(w_3) &= y_0^3 \, f^{33} \, y_1^3 \, c_3 \left(y_1^1\right)^{-1} f^{13} \left(y_0^1\right)^{-1}. \end{aligned}$$

Denote by $\bar{c}_k, \bar{y}_\ell^k, \bar{f}^{k\ell}$ the element in M obtained by substituting each of the g_i in the original words by x_i. The lengths of the $\bar{c}_k$ and $\bar{f}^{k\ell}$ are bounded, so they have finite number of possibilities. Therefore there exists a constant s_0 for which:

$$\bar{f}^{11}\bar{f}^{13} = z^{s_1}; \quad \bar{f}^{21}\bar{f}^{22} = z^{s_2}; \quad \bar{f}^{32}\bar{f}^{33} = z^{s_3}; \quad \bar{c}_1\bar{c}_2\bar{c}_3 = z^{s_4} \quad \text{and} \quad |s_i| < s_0$$

which implies

$$\lambda_T(w_1)\lambda_T(w_2)\lambda_T(w_3) = z^{\pm s_1} z^{\pm s_2} z^{\pm s_3} z^{\pm s_4} = z^t \quad \textit{and} \quad |t| < 4s_0 = q.$$

∎

Claim 4.4: U_T is Lipschitz.

Proof: Let $m_1, m_2 \in M$; $|m_1 - m_2| = 1$.

If $m_1 = m_2 z^{\pm 1}$ then $m_1 = z^{\pm\epsilon(m_2)} m_2$, $\Psi(m_1) = \Psi(m_2)$, $\epsilon(m_1) = \epsilon(m_2)$ and $|\nu(m_1) - \nu(m_2)| = 1$, so $|U_T(m_1) - U_T(m_2)| = 1$.

Otherwise, let $m_1 = m_2 x_j$. We have:

$$\left(m_2 x_1^{-\epsilon(m_2)}\right)^{-1} \left(m_1 x_1^{-\epsilon(m_1)}\right) = x_1^{\epsilon(m_2)} x_j x_1^{-\epsilon(m_1)}.$$

Therefore, by Theorem 4.1:

$$\left| \sum_{v \in V(\Psi(m_2 x_1^{-\epsilon(m_2)}))} e_v - \sum_{v \in V(\Psi(m_1 x_1^{-\epsilon(m_1)}))} e_v \right| \leq [160(\mu_0 + \delta) v_{2\delta}(20\delta + 1)]^{1/2}$$

and:

$$z^{\nu(m_2) - \nu(m_1)} = \lambda_T\left(\Psi(m_2 x_1^{-\epsilon(m_2)})\right)^{-1} \lambda_T\left(\Psi(m_1 x_1^{-\epsilon(m_1)}\right) \cdot \left(x_1^{\epsilon(m_2)} x_j x_1^{-\epsilon(m_1)}\right)^{-1}$$

which by Lemma 4.3 gives:

$$|\nu(m_2) - \nu(m_1)| \leq q$$

So:

$$|U_T(m_1) - U_T(m_2)| \leq [160(\mu_0 + \delta) v_{2\delta}(20\delta + 1)]^{1/2} + q.$$

∎

THEOREM 4.5: *U_T is uniform.*

Proof: Let $m_1, m_2 \in M$. By Lemma 4.3:

$$\lambda_T\left(\Psi(m_1 x_1^{-\epsilon(m_1)})\right) \lambda_T\left(\Psi((m_1 x_1^{-\epsilon(m_1)})^{-1} m_2 x_1^{\epsilon(m_2)})\right) = z^t \lambda_T\left(m_2 x_1^{\epsilon(m_2)}\right)$$

where $|t| < q$. Therefore:

$$\begin{aligned} |m_1 - m_2| - 2 - q &\leq |m_1 - m_2| - 2 - t \\ &\leq |\nu(m_1) - \nu(m_2)| + \left|\lambda_T\left(\Psi((m_1 x_1^{-\epsilon_1(m_1)})^{-1} m_2 x_1^{-\epsilon(m_2)})\right)\right| \end{aligned}$$

By Theorem 4.1 we have

$$\left|\lambda_T\left(\Psi((m_1x_1^{-\epsilon_1(m_1)})^{-1}m_2x_1^{-\epsilon(m_2)})\right)\right|$$

$$\leq \left|V\left(\Psi(m_1x_1^{-\epsilon_1(m_1)})\right)\triangle V\left(\Psi(m_2x_1^{-\epsilon_1(m_2)})\right)\right| + 160(\mu_0+\delta)v_{2\delta}(20\delta+1).$$

So we may conclude

$$|m_1-m_2|-2-q-160(\mu_0+\delta)v_{2\delta}(20\delta+1)\leq |U_T(m_1)-U_T(m_2)|^2.$$

∎

Remark: The referee has pointed out that the whole argument given in section 4 works if we only require the group M to be an extension of Γ by a f.g. virtually abelian group Z, such that the action of M on Z by conjugation has finite orbits. ∎

References

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